THE KERNEL AND BARGAINING SET

FOR CONVEX GAMES

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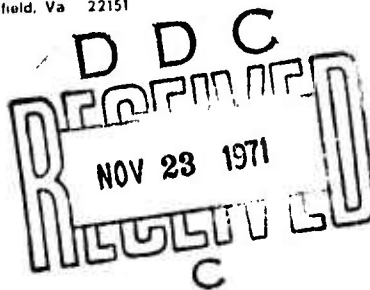
THE KERNEL AND BARGAINING SET

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1.   INTRODUCTION

Convex games were introduced in [17], where it was
shown that these are precisely the games for which the
core has a certain "regular" structure (see Section 5 be-
low). It was also shown in [17] that convex games have a
unique von-Neumann-Morgenstern solution which coincides
with the core, and that their Shapley value is essentially
the center of gravity of the extreme points of the core.

One purpose of this paper is to prove that the kernel
(for the grand coalition) of convex games consists of a

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unique point (Section 7). As such, it coincides with the

nucleolus of the game and therefore occupies a central

position in the core (which is different, in general, from

that of the Shapley value). We also prove that the bar-

gaining set $m_1^{(i)}$ (for the grand coalition) coincides with

the core (Section 8). Thus, it appears that for convex

games, many solution concepts either coincide with the core

or occupy a central position within the core.

The proofs of these results are quite elaborate and

require many lemmas drawn from various topics of game theory

(Sections 1-6). In particular, one requires a detailed

analysis of the structure of the pre-kernel of a game. We

develop this theory, which is interesting in its own sake,

in Sections 2-4, before specializing our attention to con-

vex games. The pre-kernel is related to the "pseudo kernel"

used in previous investigations [7, 8], but has the ad-

vantage of a somewhat simpler definition and is invar-

iant under strategic equivalence. If the game is stra-

tegically equivalent to a 0-normalized monotonic game

then the pre-kernel and the kernel coincide (for the

grand coalition). This is the case when the game is super-

additive, and hence, in particular, when the game is convex.

2. THE PRE-KERNEL AND ITS RELATION TO THE KERNEL AND TO THE PSEUDO KERNEL

In this section we shall introduce an auxiliary solution concept, called the pre-kernel of a game and show that if the game satisfies certain monotonicity conditions this pre-kernel coincides with the pseudo-kernel or the kernel of the game.

We shall consider a cooperative game with side payments, $(N; v)$, where $N = \{1, 2, \ldots, n\}$ is its set of players and v, its characteristic function, is an arbitrary* function from the subsets of N (called coalitions) to the real numbers.

Given an n-tuple $x = (x_1, x_2, \ldots, x_n)$ of real numbers, we define the excess of a coalition S with respect to x (in $(N; v)$) to be:

$$(2.1) \qquad e(S, x) \equiv v(S) - x(S),$$

where $x(S)$ is a short notation for $\sum_{i \in S} x_i$ whenever $S \neq \emptyset$, and $x(\emptyset) = 0$.

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*None of the traditional conditions are imposed on v at this point; in particular, $v(\emptyset)$ need not be 0.

An n-tuple $x = (x_1, x_2, \ldots, x_n)$ of real numbers will be called a <u>pre-imputation</u> (in (N; v)) if it satisfies:

(2.2) $$x(N) = v(N).$$

It will be called an <u>imputation</u> if it satisfies, in addition, the individual rationality condition:

(2.3) $$x_i \geq v(\{i\}), \quad i = 1, 2, \ldots, n.$$

It will be called a <u>pseudo-imputation</u> if it satisfies (2.2) and:

(2.4) $$x_i \geq 0, \quad i = 1, 2, \ldots, n.$$

For each n-tuple $x = (x_1, x_2, \ldots, x_n)$ we define the <u>maximum surplus</u> of a player k against a player ℓ, $k \neq \ell$, with respect to x, to be:

(2.5) $$s_{k,\ell}(x) = \underset{S:k\epsilon S, \ell\notin S}{\text{Max}} e(S, x).$$

<u>Definition 2.1</u>. A pre-imputation x is said to belong to the <u>pre-kernel</u> of a game $\Gamma \equiv (N; v)$ (for the grand coalition),* if

$$(2.6) \quad s_{k,\ell}(x) = s_{\ell,k}(x) \quad \text{for all } k, \ell \in N, k \neq \ell.$$

The pre-kernel of a game Γ (for the grand coalition) will be denoted by $Pr\mathcal{K}(\Gamma)$ or, shortly, by $Pr\mathcal{K}$.

<u>Lemma 2.2</u>. <u>The pre-kernel is a relative invariant under strategic equivalence</u>.**
    The proof is immediate.
<u>Definition 2.3</u>. A game $(N; v)$ is called <u>monotonic</u> if

$$(2.7) \quad v(S) \leq v(T) \quad \text{whenever } S \subset T.$$

Note that relation (2.7) is <u>not</u> invariant under strategic equivalence. In fact, every game is strategically equivalent to a monotonic game.***

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    *The definition can be extended to cover situations in which coalition-structures other than the grand coalition are being considered.
    **I.e., it undergoes the transformation $x \rightarrow \alpha x + a$ when $v(S)$ is replaced by $\alpha v(S) + a(S)$ for each coalition S. Here α is a real positive constant and $a = (a_1, a_2, \ldots, a_n)$ is an n-tuple of real numbers.
    ***This will be the case whenever the elements of a (see above) are sufficiently large.

Theorem 2.4. __If__ $\Gamma \equiv (N; v)$ __is strategically equivalent to a 0-normalized*__ __monotonic game and if__ $x \in Prx'(\Gamma)$ __then__ x __is an imputation__ (__see__ (2.3)).

Proof. By Lemma 2.2, there is no loss of generality in assuming that Γ is already 0-normalized. The theorem is obviously true for 1-person games. Assume that Γ has at least two players, and let $x \in Prx(\Gamma)$. It will be convenient to denote by $\mathcal{B}(x)$ the set of all coalitions of maximum excess among the coalitions other than $\emptyset$ and N:

(2.8) $\mathcal{B}(x) = \{S : S \neq \emptyset, N \text{ and } e(S, x) \geq e(R, x) \text{ whenever }$

$R \neq \emptyset, N\}$.

Suppose (2.3) is incorrect; then there exists a player k such that $x_k < 0$. We shall first show that $k \in S$ whenever $S \in \mathcal{B}(x)$. Indeed, if $k \notin S$ for some coalition S and if $S \neq N - \{k\}$ then, by (2.1) and (2.7),

$e(S \cup \{k\}, x) = v(S \cup \{k\}) - x(S) - x_k > v(S) - x(S) = e(S, x)$;

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*A game is called 0-normalized if the value of each single-person coalition is 0.

consequently, $S \notin \mathcal{B}(x)$. Also, $N - \{k\} \notin \mathcal{B}(x)$, because, by (2.2) and (2.7):

$$e(N - \{k\}, x) = v(N - \{k\}) - x(N - \{k\}) \leq v(N) - x(N) + x_k = x_k,$$

whereas, for example $e(\{k\}, x) = -x_k$ is larger.

Let R be a coalition in $\mathcal{B}(x)$ and let $\ell \in N - R$. Since k belongs to each coalition in $\mathcal{B}(x)$, it follows from (2.5) and (2.8) that $s_{k,\ell}(x) > s_{\ell,k}(x)$, contrary to (2.6). This contradiction shows that (2.3) is correct, thereby complete-ing the proof.

Definition 2.5. An imputation x (see (2.3)) is said to belong to the kernel of a game $\Gamma \equiv (N; v)$ (for the grand coalition),* if

$$(2.9) \qquad s_{k,\ell}(x) \leq s_{\ell,k}(x) \text{ or } x_\ell = v(\{\ell\}) \text{ for all}$$

$$k, \ell \in N, k \neq \ell.$$

The kernel of a game Γ (for the grand coalition) will be denoted by $\mathcal{K}(\Gamma)$ or, shortly, by $\mathcal{K}$.

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*For extension of the definition to situations in which coalition-structures other than the grand coalition are con-sidered see, e.g., [3].

Lemma 2.6. <u>The kernel is a relative invariant under stra-</u>
<u>tegic equivalence.</u>

The proof of this well known result (see, e.g., [3])
is immediate.

Theorem 2.7. <u>The kernel and the pre-kernel coincide for</u>
<u>games which are strategically equivalent to 0-normalized</u>
<u>monotonic games.</u>

<u>Proof.</u>* By Lemmas 2.2 and 2.6, we can limit the discussion
to 0-normalized games. The theorem is obviously true for
a 1-person game. Let $\Gamma \equiv (N; v)$ be an n-person 0-normalized
game, $n \geq 2$. By Theorem 2.4 and Definitions 2.1 and 2.5,
$Prk(\Gamma) \subset k(\Gamma)$. Let $x \in k(\Gamma)$; we shall complete the proof
if we show that $x \in Prk(\Gamma)$. Denote (see (2.8)).

$$(2.10) \qquad\qquad M = \cap \{S : S \in \mathcal{B}(x)\}.$$

Clearly, $M \neq N$. If $M \neq \emptyset$, let $k \in M$ and let $\ell \in N - M$.
Clearly, $s_{k,\ell}(x) > s_{\ell,k}(x)$; hence, by (2.9), $x_\ell = 0$. Let
S_0 be an arbitrary coalition in $\mathcal{B}(x)$, then, by (2.1) and
(2.7),

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*This result follows from Theorem 2.4 and from known
results stated in [8]; however, it is much more convenient
to provide here an independent, shorter proof of this im-
portant result.

$$e(S_0, x) = v(S_0) - x(S_0) = v(S_0) - x(N) \leq v(N) - x(N) = 0.$$

If $\ell \in N - S_0$ then $e(\{\ell\}, x) = 0$ and consequently $\{\ell\} \in \mathcal{B}(x)$. Thus, $M = \emptyset$, since $\mathcal{B}(x)$ contains both S_0 and $\{\ell\}$; this contradicts the assumption $M \neq \emptyset$. It follows that, in fact, $M = \emptyset$. Suppose $x \notin \mathcal{P}r\mathcal{K}(\Gamma)$ then players i and j exist such that

$$(2.11) \qquad\qquad s_{i,j}(x) > s_{j,i}(x).$$

Consequently $x_j = 0$ (see (2.9)). There exists a coalition S_1 in $\mathcal{B}(x)$ which does not contain player i, because $M = \emptyset$. Therefore, by (2.7),

$$e(S_1 \cup \{j\}, x) = v(S_1 \cup \{j\}) - x(S_1) - x_j \geq v(S_1) - x(S_1) = e(S_1, x).$$

Thus, by (2.5) and (2.8) $s_{j,i}(x)$ cannot be smaller than $s_{i,j}(x)$, contrary to (2.11). This contradiction shows that $x \in \mathcal{P}r\mathcal{K}(\Gamma)$, thereby completing the proof of the theorem.

The following remarks are intended to orient the reader who is versed with the literature, especially [7] and [8]. The proofs are similar to the proofs of Theorems 2.4 and 2.7 and will be omitted.

Remark 2.8. The pseudo-kernel (for the grand coalition) of a game $\Gamma \equiv (N; v)$ is denoted by $Ps\mathcal{K}(\Gamma)$ and defined exactly as in Definition 2.5, except that x is assumed to be a pseudo imputation (see (2.4)) and that $x_\ell = v(\{\ell\})$ in (2.9) is replaced by $x_\ell = 0$. It is an auxiliary solution concept which is <u>not</u> a relative invariant under strategic equivalence. If Γ satisfies

(2.12) $v(S) \leq v(T)$ whenever $S \subset T$, $S \neq \emptyset$, $T \neq N$

(quasi monotonicity), and

(2.13) $v(\{i\}) + v(N) \geq v(N - \{i\})$,   i = 1, 2, ..., n,

and if $x \in Pr\mathcal{K}(\Gamma)$, then x is a pseudo-imputation. Moreover, under the conditions (2.12) - (2.13), $Pr\mathcal{K}(\Gamma) = Ps\mathcal{K}(\Gamma)$.

Remark 2.9. We can interpret the pre-kernel of a game Γ as follows: Take a game Γ^* which is monotonic, satisfies $v(\emptyset) \geq 0$, and is strategically equivalent to Γ. The "inverse image" of the pseudo-kernel of Γ^* under this equivalence is the pre-kernel of Γ. Thus, loosely speaking, up to strategic equivalence, the pre-kernel is one of many pseudo-kernels a game may have.

Remark 2.10. Since the pseudo-kernel of a game is not empty
if $v(N) \geq 0$ (see [7], [8]), it follows that the pre-kernel
of any cooperative game is not empty.

3. THE STRUCTURE OF THE PRE-KERNEL

Let x be a pre-imputation in a game $\Gamma \equiv (N; v)$. We wish to find necessary and sufficient conditions that $x \in Pr\kappa(\Gamma)$. First, let us partition the set of all the coalitions into subsets $e^1(x)$, $e^2(x)$, ..., $e^m(x)$ which are of highest excess, of the second highest excess, etc. Thus,

(3.1) $e^1(x) \equiv \{S : e(S, x) \geq e(T, x) \text{ all } T\}$,

(3.2) $e^{i+1}(x) \equiv \{S : e(S, x) \geq e(T, x)$

$$\text{if and only if } T \notin \bigcup_{h=1}^{i} e^h(x)\};$$

and $m = m(x)$ is the highest index i for which $e^i(x) \neq \emptyset$. Clearly, $1 \leq m \leq 2^n$.

We shall refer to the coalitions in $e^i(x)$ as the i-th stage maximum excess coalitions. Their excess $s^i(x)$ will be called the i-th stage maximum excess:

(3.3) $s^i(x) = e(S, x) \text{ where } S \in e^i(x)$.

Denote:

(3.4) $i(k, \ell, x) \equiv \text{Min } \{i : \exists S \in \mathcal{C}^i(x), k \in S, \ell \notin S\};$

then, clearly (see (2.5)),

(3.5) $s_{k,\ell}(x) = s^{i(k,\ell,x)}(x).$

The following lemma follows from Definition 2.1 and
(3.5).

Lemma 3.1. A pre-imputation x belongs to $P_{rK}(\Gamma)$ if and
only if $i(k, \ell, x) = i(\ell, k, x)$ for each pair of distinct
players k and ℓ.

We can now reverse the procedure. Consider an arbitrary
vectorial partition* $(\mathcal{C}^1, \mathcal{C}^2, \ldots, \mathcal{C}^m)$ of the set of all
coalitions which has the property:

(3.6) $i(k, \ell) = i(\ell, k)$ for all $\ell, k \in N, \ell \neq k,$

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*It is important to distinguish the stages. Thus,
for N = {1, 2, 3}, we consider ({∅, N}, {{1, 2}, {1, 3},
{2, 3}}, {{1}, {2}, {3}}) to be different from ({∅, N},
{{1}, {2}, {3}}, {{1, 2}, {1, 3}, {2, 3}}). For this
reason we use the vector notation and call the partition
vectorial.

where

(3.7) $i(k, \ell) \equiv \text{Min } \{i : \exists S \in \mathcal{C}^i, k \in S, \ell \notin S\}.$

Every pre-imputation x satisfying

(3.8) $\mathcal{C}^i(x) = \mathcal{C}^i, \text{ i } i = 1, 2, \ldots, m,$

must belong to $\mathcal{P}r\mathcal{X}(\Gamma)$.

Observe that the set of pre-imputations satisfying (3.8) for a fixed vectorial partition is a (possibly empty) convex set determined by the linear inequalities:

$$(3.9) \quad \begin{cases} x(N) = v(N) \\ e(S, x) > e(T, x) \text{ whenever } S \in \mathcal{C}^\mu, T \in \mathcal{C}^\nu, \mu < \nu \\ e(S, x) = e(T, x) \text{ whenever } S, T \in \mathcal{C}^\mu. \end{cases}$$

Our next object is to find conditions which assure us that a vectorial partition satisfies (3.6). The following definition is helpful:

Let $\mathcal{C}$ be a collection of subsets of N and let T be a non-empty subset of N. Let $\{T_1, T_2, \ldots, T_\alpha\}$ be the partition of T characterized by:

(3.10) $k, \ell \in T_j \Leftrightarrow (k, \ell \in T$ and $k \in A$ if and only if

$\ell \in A$ for all $A \in \mathcal{C})$.

<u>Definition 3.2</u>. The set $\{T_1, T_2, \ldots, T_\alpha\}$ defined by
(3.10) will be called <u>the partition of</u> T <u>into equivalence</u>
<u>classes induced by</u> $\mathcal{C}$.

Equivalence classes in this connection mean equiva-
lence classes determined by the relation "occur simultan-
eously in the coalitions of $\mathcal{C}$".

Let $\underset{\sim}{E} \equiv (\mathcal{C}^1, \mathcal{C}^2, \ldots, \mathcal{C}^m)$ be an arbitrary vectorial
partition of the set of coalitions. We shall now construct
a sequence of successively finer partitions of N, called
the <u>profile</u> $P(\underset{\sim}{E})$ <u>generated by</u> $\underset{\sim}{E}$.

We start by denoting $\{N\}$ as $\{T_1^1\}$. Suppose that
$\{T_1^i, T_2^i, \ldots, T_{u_i}^i\}$ has been defined, and is a partition of N.
Let $\{T_{j,1}^{i+1}, T_{j,2}^{i+1}, \ldots, T_{j,\alpha_j}^{i+1}\}$ be the set of equivalence classes
which are induced by $\mathcal{C}^i$ on T_j^i, $j = 1, 2, \ldots, u_i$. Renumber
$T_{j,\cdot}^{i+1}$ lexicographically in the lower indices to form
$\{T_1^{i+1}, T_2^{i+1}, \ldots, T_{u_{i+1}}^{i+1}\}$. The collection $P(\underset{\sim}{E}) \equiv$
$\{T_1^1; T_1^2, \ldots, T_{u_2}^2; \ldots; T_1^{m+1}, \ldots, T_{u_{m+1}}^{m+1}\}$ is the required
profile. The term is suggested by the diagram below.

$$T_1^1 \equiv N$$

Clearly,

$$(3.11) \quad \{T_1^{m+1}, T_2^{m+1}, \ldots, T_{u_{m+1}}^{m+1}\} = \{\{1\}, \{2\}, \ldots, \{n\}\},$$

but in general the equivalence classes may all become 1-person sets at an earlier stage. The next three lemmas follow directly from the definitions.

Lemma 3.3.   If $1 \le i_0 \le i_1 \le m + 1$ then

$$(3.12) \quad T_{j_0}^{i_0} \cap T_{j_1}^{i_1} \ne \emptyset \quad \text{implies} \quad T_{j_1}^{i_1} \subset T_{j_0}^{i_0}.$$

Lemma 3.4.   If $S \in \ell^i$ then S is a union of sets T_j^{i+1}'s.

Lemma 3.5. If $S \in \mathcal{C}^i$ then S <u>is a union of sets</u> $T_j^{i_1}$'s <u>when-</u>
<u>ever</u> $i < i_1 \leq m + 1$.

Henceforth, the profile $P(\underset{\sim}{E}(x))$ generated by the par-
tition $\underset{\sim}{E}(x) \equiv (\mathcal{C}^1(x),\ \mathcal{C}^2(x),\ \ldots,\ \mathcal{C}^m(x))$ will be called,
shortly, <u>the profile of</u> x.

Lemmas 3.3-3.4 indicate that the profile can be de-
scribed as a "partition tree"; namely, as a tree whose
vertices are the sets T_j^i, with $T_1^1 = N$ the root, such that
the vertices that follow a vertex T_j^i and are adjacent to
it form a partition of T_j^i.

One of the advantages of the profile of a vectorial
partition is the fact that it enables one to describe con-
dition (3.6) in a more visual fashion:

Lemma 3.6. <u>Let</u> $P(\underset{\sim}{E})$ <u>be a profile generated by a vectorial</u>
<u>partition</u> $\underset{\sim}{E} \equiv (\mathcal{C}^1,\ \mathcal{C}^2,\ \ldots,\ \mathcal{C}^m)$. <u>The condition</u> (3.6) <u>is</u>
<u>equivalent to the following separation condition:</u>

<u>If</u> $T_k^{i+1} \subset T_j^i$, $T_\ell^{i+1} \subset T_j^i$ <u>and</u> $k \neq \ell$, <u>then there</u>
<u>exists a coalition</u> S <u>in</u> $\mathcal{C}^i$ <u>such that</u> $T_k^{i+1} \subset S$ <u>and</u>
$T_\ell^{i+1} \cap S = \emptyset$, $i = 1, 2, 3, \ldots, m$.

<u>Proof</u>. Name a player in T_k^{i+1} and a player in T_ℓ^{i+1} by k

and ℓ, respectively. Then by Lemma 3.3, k, ℓ belong to

the same equivalence class $T_{\nu_i}^{i'}$ for each i' $\leq$ i. They belong

to disjoint equivalence classes T_k^{i+1} and T_ℓ^{i+1}. By Defini-

tion 3.2,

    (i)   k $\in$ A $\Leftrightarrow$ ℓ $\in$ A whenever A $\in$ $\mathcal{C}^{i'}$ and i' $\leq$ i - 1

and either

    (ii) $\exists$S $\in$ $\mathcal{C}^i$ such that k $\in$ S and ℓ $\notin$ S,

or

    (iii) $\exists$S $\in$ $\mathcal{C}^i$ such that ℓ $\in$ S and k $\notin$ S

(or both). Now

    (i) and (ii) $\Rightarrow$ i(k, ℓ) = i,

    (i) and (iii) $\Rightarrow$ i(ℓ, k) = i.

It follows that condition (3.6) is equivalent to the va-

lidity of (i), (ii), and (iii) for i = 1, ..., m and for

all k, ℓ $\in$ N, k $\neq$ ℓ.

    By Lemma 3.1 (see also (3.8)), and Lemma 3.6, we can

now state:

<u>Theorem 3.7</u>. <u>Let</u> x <u>be a pre-imputation in a game</u> Γ <u>and</u>

<u>let</u> $P(\underset{\sim}{E}(x))$ <u>be the profile of</u> x. <u>With this notation,</u>

x $\in$ $Prk(\Gamma)$ <u>if and only if the separation condition in</u>

<u>Lemma</u> 3.6 <u>is satisfied, with</u> $\mathcal{C}^i = \mathcal{C}^i(x)$, i = 1, 2, ..., m.

4. THE STAGE GAMES

From a visual point of view, a profile may contain
smaller profiles. The figure below exhibits one profile
within the original one. This suggests that smaller games
can be constructed from the original game, which contain
fewer players. Such games can serve for induction pur-
poses.
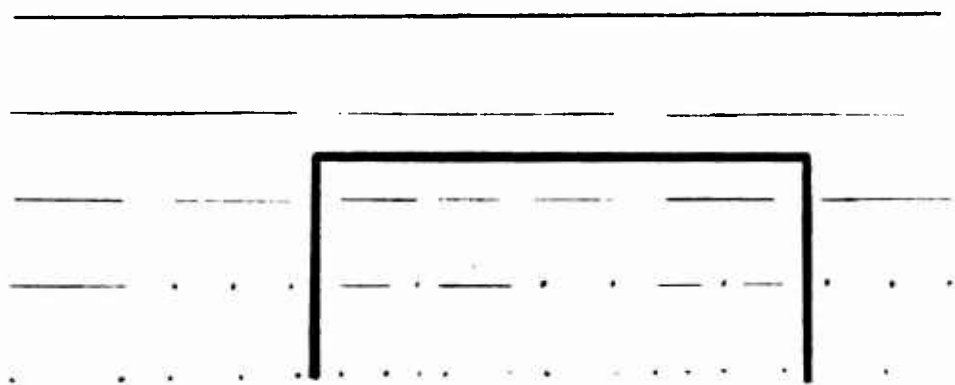
Theorem 3.7 indicates that the equivalence classes play
a role at each stage, rather than the players. Even the max-
imum excess coalitions of the various stages are unions of
such equivalence classes (Lemma 3.4). This suggests that
it is possible under an appropriate interpretation to regard
the equivalence classes themselves as players in some sense.
In the present section we shall develop these heuristic ideas
in a precise way.

<u>Definition 4.1</u>. Let x be a pre-imputation in a game $\Gamma \equiv$ $(N; v)$ and let $P(\underset{\sim}{E}(x)) = \{T_1^1; T_1^2, \ldots, T_{u_2}^2; \ldots;$ $T_1^{m+1}, \ldots, T_{u_{m+1}}^{m+1}\}$ be the profile of x. Let $T^* =$ $\{T_{j_1}^i, T_{j_2}^i, \ldots, T_{j_\alpha}^i\}$ be a fixed nonempty set of equivalence classes belonging to a fixed stage i. The <u>stage</u> <u>game generated by</u> x <u>and</u> T^* is a game $(T^*; v^*)$ whose players are the members of T^* and whose characteristic function is defined by

$$(4.1) \quad \begin{cases} v^*(T^*) = x(T_{j_1}^i) + x(T_{j_2}^i) + \ldots + x(T_{j_\alpha}^i) = x(T) \\ \\ v^*(S^*) = \underset{Q:Q \subset N-T}{Max} \ [v(S \cup Q) - x(Q)], \ S^* \subset T^*, \ S^* \neq T^*. \end{cases}$$

Here, $T \equiv T_{j_1}^i \cup T_{j_2}^i \cup \ldots \cup T_{j_\alpha}^i$ and if $S^* = \{T_{\nu_1}^i, T_{\nu_2}^i, \ldots, T_{\nu_\beta}^i\} \subset T^*$, then $S = T_{\nu_1}^i \cup T_{\nu_2}^i \cup \ldots \cup T_{\nu_\beta}^i$.

<u>Remark 4.2</u>. Note that $(x(T_{j_1}^i), x(T_{j_2}^i), \ldots, x(T_{j_\alpha}^i))$ is a pre-imputation in the above stage game.

<u>Definition 4.3</u>. A pre-imputation x in a game Γ is said to belong to the <u>core</u> of Γ if

$$(4.2) \qquad e(S, x) \leq 0 \qquad \text{all } S.$$

The core will be denoted by $\mathcal{C}(\Gamma)$ or, shortly, $\mathcal{C}$.

Remark 4.4. If $x \in \mathcal{C}(\Gamma)$ then x is an imputation (see (2.3)).

Proof. Individual rationality is nothing but (4.2) applied to single-person coalitions.

Lemma 4.5. If $\Gamma \equiv (N; v)$ is a monotonic game (see Definition 2.3) and if $x \in \mathcal{C}(\Gamma)$ then the stage game $(T^*; v^*)$ generated by x and T^* (see Defnition 4.1) is also a monotonic game.

Proof. Quasi monotonicity (see (2.12)) follows directly from (4.1) and the monotonicity of Γ. (We even make no use of the fact that $x \in \mathcal{C}(\Gamma)$.) Let $S^* \subset T^*$, $S^* \neq T^*$, then, by (4.1), there exists a subset Q_0 of N - T such that $v^*(S^*) = v(S \cup Q_0) - x(Q_0)$. Thus, by (4.1) and (4.2), $v^*(S^*) - v^*(T^*) = v(S \cup Q_0) - x(Q_0) - x(T) \leq v(T \cup Q_0) - x(T \cup Q_0) \leq 0$, and this concludes the proof.

We are now in a position to state the main theorem of this section:

Theorem 4.6. If $x \in \mathcal{P}r\mathcal{K}(\Gamma)$ and if $\Gamma^* \equiv (T^*; v^*)$ is a stage game generated by x and a set $T^* = \{T^i_{j_1}, T^i_{j_2}, \ldots, T^i_{j_\alpha}\}$ of equivalence classes of the i-th stage, $1 \leq i \leq m + 1$, then

the α-tuple $x^* \equiv (x(T^i_{j_1}), x(T^i_{j_2}), \ldots, x(T^i_{j_\alpha}))$ belongs to $\text{Pr}\varkappa(\Gamma^*)$.

Proof. We shall use stars to denote entities related to Γ^*. By Remark 4.2, x^* is a pre-imputation in Γ^*; consequently, there is nothing more to prove if $\alpha = 1$. Suppose $\alpha > 1$. We have to show that x^* satisfies the analogue of (2.6):

$$(4.3) \qquad s^*_{T^i_\rho, T^i_\sigma}(x^*) = s^*_{T^i_\sigma, T^i_\rho}(x^*)$$

for all "stage players" T^i_ρ, $T^i_\sigma \in T^*$, $\rho \neq \sigma$. Here,

$$(4.4) \quad s^*_{T^i_\rho, T^i_\sigma}(x^*) \equiv \text{Max } \{e^*(S^*, x^*) : S^* \subset T^*, T^i_\rho \in S^*,$$
$$T^i_\sigma \notin S^*\}$$

and

$$(4.5) \quad e^*(S^*, x^*) \equiv v^*(S^*) - x^*(S^*) = v^*(S^*) - x(S)$$

where, as in (4.1), S is defined as $T^i_{\nu_1} \cup T^i_{\nu_2} \cup \ldots \cup T^i_{\nu_\beta}$ if $S^* = \{T^i_{\nu_1}, T^i_{\nu_2}, \ldots, T^i_{\nu_\beta}\}$ with $\{\nu_1, \nu_2, \ldots, \nu_\beta\} =$ $\{j_1, j_2, \ldots, j_\alpha\}$. By (4.4), (4.5), and (4.1),

$$s^*_{T^i_\rho, T^i_\sigma}(x^*) = \text{Max} \left\{ \begin{array}{c} \text{Max} \\ Q:Q \subset N-T \end{array} e(S \cup Q, x) : S^* \subset T^*, T^i_\rho \in S^*, \right.$$

$$\left. T^i_\sigma \notin S^* \right\}.$$

where k is any player in T^i_ρ and l is any player in T^i_σ. We assert that in fact

$$(4.6) \quad s^*_{T^i_\rho, T^i_\sigma}(x^*) = \text{Max} \{e(R, x) : R \subset N, k \in R, l \notin R\}$$

$$\equiv s_{k,l}(x).$$

The argument for this runs as follows: A priori, there should be an inequality $\leq$, because the set of candidates for maximization increases. It is known however that $s_{k,l}(x) = s^{i(k,l,x)}(x)$ (see (3.5)). Since T^i_ρ and T^i_σ are distinct equivalence classes of the i-th stage, it follows that $i(k, l, x) \leq i - 1$ (see (3.4)). Let R^0 be a coalition containing k and not l such that $s_{k,l}(x) = e(R^0, x)$; then $R^0 \in \wp^{i(k,l,x)}(x)$ (see 3.5)). Since $i(k, l, x) \leq i - 1$, it follows from Lemma 3.5 that R^0 is a union of equivalence classes of the i-th stage and, moreover, $R^0 \supset T^i_\rho$ and $R^0 \cap T^i_\sigma = \emptyset$. Thus, R^0 has the form $T^i_{\nu_1} \cup T^i_{\nu_2} \cup \ldots \cup T^i_{\nu_\beta} \cup Q$,

where $\{v_1, v_2, \ldots, v_\beta\}$ is a subset of $\{j_1, j_2, \ldots, j_\alpha\}$ containing ρ and not σ, and $Q \subset N - T$ It is therefore a member of the smaller set of candidates, which proves (4.6). In a similar fashion we prove that $s^*_{T^i_\sigma, T^i_\rho}(x^*) = s_{\ell, k}(x)$. Since $x \in Pr\mathcal{K}(\Gamma)$, (4.3) now follows from (2.6).

Remark 4.7. A converse theorem stating that if $x^* \in Pr\mathcal{K}(\Gamma^*)$ for each stage game then $x \in Pr\mathcal{K}(\Gamma)$ is trivially true, because the stage game $(T^*; v^*)$ where T^* is the set of all equivalence classes of the stage $m + 1$ is isomorphic to Γ under the transformation $\{k\} \to k$, $k = 1, 2, \ldots, n$ (see (3.11)).

Remark 4.8. Theorem 4.6 generalizes results of[*] [8]. The stage game in which T^* consists of all the equivalence classes of a given stage is known as the intermediate game. The stage game in which $i = m + 1$ and $\{j_1, j_2, \ldots, j_\alpha\}$ are players of a given T^i_j is known as a reduced game.

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[*]The results in [8], however, refer to the wider class of pseudo-kernels (see Remark 2.9).

5. THE STAGE GAMES RESULTING FROM AN IMPUTATION IN THE CORE OF A CONVEX GAME

A cooperative game (N; v) is called <u>convex</u> if its characteristic function v satisfies

(5.1) $v(\emptyset) = 0$,

(5.2) $v(A) + v(B) \leq (A \cup B) + v(A \cap B)$ all A, B $\subset$ N.

Convex games were introduced in [17], where their properties and their importance in game theory were discussed. At present, all we need to know of their properties, beyond (5.1) and (5.2), is that they have nonempty cores.*

The purpose of this section is to show that for an x in the core of a convex game, all the stage games are also convex. We shall also study some properties of these stage games.

Convex games are super-additive but not necessarily monotonic. However, if the characteristic function satisfies

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*It is proved in [17] that they can be characterized by the fact that their core is, so called, <u>regular</u>--i.e., for each x in the core, the family $S_x = \{S : x(S) = v(S)\}$ is closed under union and intersection. (Compare Lemma 6.2 below.)

(5.3) $v(\{i\}) \geq 0,$ $i = 1, 2, \ldots, n,$

then monotonicity follows from super-additivity. Since
being a convex game is an invariant under strategic equi-
valence, it follows that convex games are strategically
equivalent to 0-normalized monotonic games. In view of
Theorem 2.7 and Remark 2.8, we can therefore state:

Theorem 5.1. <u>If Γ is a convex game then</u>

(5.4) $\varkappa(\Gamma) = \mathcal{P}r\varkappa(\Gamma).$

<u>If Γ is a convex game with a nonnegative characteristic</u>
<u>function then</u>

(5.5) $\varkappa(\Gamma) = \mathcal{P}s\varkappa(\Gamma).$

Note that (5.2) is equivalent to

(5.6) $e(A, x) + e(B, x) \leq e(A \cup B, x) + e(A \cap B, x)$

for all A, B $\subset$ N and for any n-tuple x.

Theorem 5.2. If $\Gamma \equiv (N; v)$ <u>is a convex game and if x be-</u>
<u>longs to its core, then each stage game generated by x is</u>
<u>convex</u>.

<u>Proof</u>. Let $(T^*; v^*)$ be a stage game generated by x and $T^* = \{T^i_{j_1}, T^i_{j_2}, \ldots, T^i_{j_\alpha}\}$. We shall show that

(5.7) $v^*(S^*) + v^*(R^*) \le v^*(S^* \cup R^*) + v^*(S^* \cap R^*)$

         all S^*, $R^* \subset T^*$.

Relation (5.7) evidently holds if $S^* \subset R^*$ or if $R^* \subset S^*$.
We can therefore assume that S^*, $R^* \neq T^*$. Let S and R be
the unions of the members of S^* and R^*, respectively. By
(4.1), there exists Q_1 and Q_2 in N - T such that

$v^*(S^*) + v^*(R^*) = v(S \cup Q_1) - x(Q_1) + v(R \cup Q_2) - x(Q_2)$

$\le v((S \cup R) \cup (Q_1 \cup Q_2)) + v((S \cap R) \cup (Q_1 \cap Q_2))$

$- x(Q_1 \cup Q_2) - x(Q_1 \cap Q_2)$

$\le \underset{Q:Q \subset N-T}{\text{Max}} \quad [v((S \cup R) \cup Q) - x(Q)]$

$+ \underset{Q:Q \subset N-T}{\text{Max}} \quad [v((S \cap R) \cup Q) - x(Q)]$

$= \underset{Q:Q \subset N-T}{\text{Max}} \quad [v((S \cup R) \cup Q) - x(Q)] + v^*(S^* \cap R^*).$

If $S^* \cup R^* = T^*$, then

$$\underset{Q:Q \subseteq N-T}{\text{Max}} \quad [v((S \cup R) \cup Q) - x(Q)] \leq v^*(T^*) = v^*(S^* \cup R^*),$$

because $x \in \mathcal{C}(\Gamma)$ (see Lemma 4.5). If $S^* \cup R^* \neq T^*$, then, by (4.1),

$$\underset{Q:Q \subseteq N-T}{\text{Max}} \quad [v((S \cup R) \cup Q) - x(Q)] = v^*(S^* \cup R^*),.$$

In any case (5.7) holds.

The following lemma furnishes important information concerning the particular Q's for which the maxima in (4.1) are achieved, when the game is convex.

Lemma 5.3. Let $\Gamma \equiv (N; v)$ be a convex game and let x be an arbitrary n-tuple of real numbers. Let R be a coalition in $e^i(x)$ and let S_1 and S_2 be subsets of R and $N - R$, respectively. Suppose Q_1 and Q_2 are subsets of $N - R$ and R, respectively, such that

(5.8) $$\underset{Q:Q \subseteq N-R}{\text{Max}} \quad e(S_1 \cup Q, x) = e(S_1 \cup Q_1, x),$$
and

(5.9) $$\underset{Q:Q \subseteq R}{\text{Max}} \quad e(S_2 \cup Q, x) = e(S_2 \cup Q_2, x).$$

<u>Let</u> $R \cup Q_1$ <u>and</u> Q_2 <u>belong to</u> $\mathcal{C}^{\mu_1}(x)$ <u>and</u> $\mathcal{C}^{\mu_2}(x)$, <u>respectively</u>.

<u>Under these conditions</u>:

    (i) $\mu_1 \leq i$,

    (ii) $\mu_2 \leq i$,

    (iii) <u>If</u> $e(S_1 \cup Q_1, x) \neq e(S_1, x)$ <u>then</u> $\mu_1 < i$,

    (iv)   <u>If</u> $e(S_2 \cup Q_2, x) \neq e(S_2 \cup R, x)$ <u>then</u> $\mu_2 < i$.

<u>Proof</u>. By (5.6),

(5.10) $e(S_1 \cup Q_1, x) + e(R, x) \leq e(R \cup Q_1, x) + e(S_1, x)$.

By (5.8), $e(S_1 \cup Q_1, x) \geq e(S_1, x)$. Consequently,

(5.11) $e(R, x) \leq e(R \cup Q_1, x)$,

and strict inequality holds if the hypothesis of (iii) is satisfied. This proves (i) and (iii) (see (3.1), (3.2)). Similarly, by (5.6),

(5.12) $e(S_2 \cup Q_2, x) + e(R, x) \leq e(S_2 \cup R, x) + e(Q_2, x)$.

By (5.9), $e(S_2 \cup Q_2, x) \geq e(S_2 \cup R, x)$. Consequently,

$$(5.13) \qquad e(R, x) \leq e(Q_2, x),$$

and strict inequality holds if the hypothesis of (iv) is satisfied. This proves (ii) and (iv).

Corollary 5.4. Q_1 and Q_2 of Lemma 5.3 can be chosen to be unions of equivalence classes of stage $i + 1$ in the profile of x.

Proof. Cases (i) and (ii) of Lemma 5.3 and Lemma 3.5.

Corollary 5.5. If $R \in \mathcal{e}^1(x)$ and Γ is convex then

$$(5.14) \quad \underset{Q:Q \subset N-R}{\text{Max}} \quad e(S \cup Q, x) = e(S, x) \text{ whenever } S \subset R,$$

$$(5.15) \quad \underset{Q:Q \subset R}{\text{Max}} \quad e(S \cup Q, x) = e(S \cup R, x) \text{ whenever } S \subset N - R.$$

Proof. Cases (iii) and (iv) of Lemma 5.3 (see (3.1)).

Lemma 5.3 can be effectively used in devising computer programs for computing the kernels of convex games. Note

that it can be applied to any stage game (T*; v*) of a
stage greater than i, when the union of the members of T*
is equal to R. We shall subsequently apply Lemma 5.3
for the particular cases i = 1, 2 and the stage game being
of stage m + 1.

Lemma 5.6. Let x be a pre-imputation in a game $\Gamma \equiv (N; v)$
satisfying $v(\emptyset) = 0$. Under these conditions, exactly one
of the following relations holds:

$\quad$ (i) $\quad \mathcal{B}(x) = e^1(x)$,

$\quad$ (ii) $\quad \mathcal{B}(x) \cup \{\emptyset, N\} = e^1(x)$,

$\quad$ (iii) $\quad \mathcal{B}(x) = e^2(x)$ and $e^1(x) = \{\emptyset, N\}$.

If $x \notin \mathcal{C}(\Gamma)$, case (i) holds, and if $x \in \mathcal{C}(\Gamma)$, case (ii)
or case (iii) holds.

Proof. Compare (2.8) with (3.1) and (3.2) and the defini-
tion of the core.

Lemma 5.7. If x belongs to the core of a convex game
$\Gamma \equiv (N; v)$, and if $R \in \mathcal{B}(x)$, then

$$(5.16) \quad \underset{Q:Q \subset N-R}{\text{Max}} \; e(S \cup Q, x) = \text{Max}[e(S, x), e(S \cup (N - R), x)]$$

whenever $S \subset R$, and

(5.17) $\underset{Q:Q\subset R}{\text{Max}}$ $e(S \cup Q, x) = \text{Max} [e(S, x), e(S \cup R, x)]$

whenever $S \subset N - R$.

Proof.   Corollary 5.5, if $\mathcal{B}(x) \subset \mathcal{C}^1(x)$.   If this is not

the case then, by Lemma 5.6, $\mathcal{C}^1(x) = \{\emptyset, N\}$ and $\mathcal{B}(x) = \mathcal{C}^2(x)$.

The result now follows from Lemma 5.3, cases (iii) and (iv).

Corollary 5.8.   Let x belong to the core of a convex game

and let R be a coalition in $\mathcal{B}(x)$ (see (2.8)).   Consider

the stage games (T^*_R, v^*_R) and (T^*_{N-R}, v^*_{N-R}) of any stage i,

such that the union of the members of T^*_R is equal to R

and the union of the members of T^*_{N-R} is equal to N - R.

Under these conditions

(5.18) $\begin{cases} v^*_R(T^*_R) = x(R) \\ \\ v^*_R(S^*) = \text{Max} [v(S), v(S \cup (N - R)) - x(N - R)] \end{cases}$

whenever $S^* \subset T^*_R$, $S^* \neq T^*_R$, and

(5.19) $\begin{cases} v^*_{N-R}(T^*_{N-R}) = x(N - R) \\ \\ v^*_{N-R}(S^*) = \text{Max} [v(S), v(S \cup R) - x(R)] \end{cases}$

whenever $S^* \subset T^*_{N-R}$, $S^* \neq T^*_{N-R}$. Here, as before, S is de-
fined as $T^i_{\mu_1} \cup T^i_{\mu_2} \cup \ldots \cup T^i_{\mu_\beta}$ if $S^* = \{T^i_{\mu_1}, T^i_{\mu_2}, \ldots, T^i_{\mu_\beta}\}$.

In other words, the values of v^*_R and v^*_{N-R} are reduced

to two possibilities when Γ is convex, $x \in \mathcal{C}(\Gamma)$, and $R \in \mathcal{B}(x)$.

6. COMPLETELY SEPARATING NEAR-RING COLLECTIONS

<u>Definition 6.1</u>. A collection $\mathcal{C}$ of subsets of a set N is called a <u>near-ring</u>* if:

$$A \cup B = N, \text{ or}$$

(6.1) $A, B \in \mathcal{C} \Rightarrow$ $A \cap B = \emptyset$, or

both $A \cup B \in \mathcal{C}$ and $A \cap B \in \mathcal{C}$.

<u>Lemma 6.2</u>. <u>If Γ is a convex game and x is an arbitrary n-tuple of real numbers, then $\mathcal{B}(x)$ (see (2.8)) is a near-ring</u>.

<u>Proof</u>. Combine (5.6) with (2.8).

<u>Definition 6.3</u>. A collection $\mathcal{C}$ of subsets of a set N is said to be <u>completely separating</u> (over N) if for each ordered pair (k, ℓ) of distinct elements of N there exists a set in $\mathcal{C}$ containing k and not ℓ.

<u>Definition 6.4</u>. A collection $\mathcal{C}$ of subsets of a set N is called <u>separating</u> (over N) if for each ordered pair (k, ℓ) of distinct elements of N, whenever a coalition exists in $\mathcal{C}$

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*We are grateful to J. R. Isbell for suggesting this term.

that contains k and not ℓ, another coalition exists in $\mathcal{C}$
that contains ℓ and not k.

Let $\mathcal{C}$ be a separating collection of subsets of N.
Let T_1, T_2, ..., T_u be the equivalence classes induced by
$\mathcal{C}$ on N. Let $N^\sim$ be a subset of N containing exactly one
member from each equivalence class. Clearly, the collec-
tion $\mathcal{C}^\sim \equiv \{S \cap N^\sim : S \in \mathcal{C}\}$ is completely separating over
$N^\sim$.

The study of the separating and the completely sep-
arating collections has been quite useful to kernel theory
(see, e.g., [7]).* In fact, the separation condition in
Lemma 3.6 simply states that the set $\{S \cap T_j^i : S \in \mathcal{C}^i\}$ is
separating over the equivalence class T_j^i. A particular
case of this observation is:

Lemma 6.5. If $x \in \mathcal{P}r\mathcal{K}(\Gamma)$ then $\mathcal{B}(x)$ is a separating collec-
tion.

Proof. Theorem 3.7 and Lemma 5.6, in view of the fact that
deleting the coalitions $\emptyset$ and N from a collection of coali-
tions does not change its being or not being separating.

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*See [12] and [15] for additional properties of sep-
arating collections.

It will be convenient to associate with a subset S of
N its characteristic vector χ^S, where

(6.2)
$$\chi_i^S = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$$

Definition 6.6. A collection $\mathcal{C} = \{S_1, S_2, \ldots, S_\alpha\}$ of
subsets of a set N is called balanced, if positive con-
stants $c_1, c_2, \ldots, c_\alpha$ exist, such that

(6.3)
$$\sum_{\nu=1}^{\alpha} c_\nu \chi^{S_\nu} = \chi^N.$$

$\mathcal{C}$ is called minimal balanced if it is balanced and none
of its proper sub-collections is balanced. $\mathcal{C}$ is called
weakly balanced if (6.3) is satisfied by nonnegative con-
stants $c_1, c_2, \ldots, c_\alpha$. These constants are called balanc-
ing coefficients or weights.

Balanced and minimal balanced collections were intro-
duced* and studied in [2] and [16]. They are useful to

*O. N. Bondareva [2] uses the term "(q-θ)-covering"
["reduced (q-θ-covering"] to denote the pair consisting
of the set of weights and the set of characteristic vectors
of a balanced [minimal balanced] collection. Sometimes,
(e.g., in [13]) "balanced" means what we here call "weakly
balanced", however, note that every weakly balanced collec-
tion contains a balanced collection. It was convenient in
[16] to rule out the collection {N}; this exception is not
needed here.

the study of various solution concepts such as the core
(see [2], [13], [16], [18]), the bargaining set (see [6]),
and, as we shall see here, the kernel. See [11] for addi-
tional information concerning their structure.

Lemma 6.7. A balanced collection is separating.

The proof is straightforward. The converse statement,
however, is not true. Indeed, any set of six minimal
winning coalitions in the 7-person projective game (see,
e.g. [19], p. 470) is completely separating and not even
weakly balanced. It turns out, however, that imposing a
near-ring requirement (see Definition 6.1) is a remedy:

Theorem 6.8. Every separating near-ring collection $\mathcal{C}$ of
subsets of a set $N = \{1, 2, \ldots, n\}$, with the exception of
$\mathcal{C} = \{\emptyset\}$, is weakly balanced.

Proof. There is no loss of generality in assuming that $\mathcal{C}$
is completely separating. The theorem obviously holds for
$n = 1$. Assume $n \geq 2$. Let $\mathcal{C}_i \equiv \{S : S \in \mathcal{C}, i \notin S\}$ and let
$\widetilde{\mathcal{C}_i}$ denote the set of elements of $\mathcal{C}_i$ which are maximal under
inclusion. We shall show that $\widetilde{\mathcal{C}_i}$ is a partition of $N - \{i\}$.
Indeed, it follows from the complete separating property
that each member of $N - \{i\}$ belongs to at least one element

of $\widetilde{e_i}$, and by the near-ring property, the elements of $\widetilde{e_i}$ are disjoint. We next observe that the collection $\widetilde{e} \equiv \widetilde{e_1} \cup \widetilde{e_2} \cup \ldots \cup \widetilde{e_n}$ is balanced; in fact, if $c(S)$ is the number of elements i such that $S \in \widetilde{e_i}$, then $\{c(S)/(n-1) : S \in \widetilde{e}\}$ are balancing coefficients. Hence $e \supset \widetilde{e}$ is at least weakly balanced. This completes the proof.

7. THE KERNEL OF A CONVEX GAME

The purpose of this section is to show that the kernel of a convex game (for the grand coalition) consists of a single point.

Lemma 7.1. If Γ is a convex game then $\mathcal{P}r\mathcal{K}(\Gamma) \subset \mathcal{C}(\Gamma)$.

Proof. The theorem obviously holds if Γ is a one-person game. Assume that Γ is a multi-person game and let $x \in \mathcal{P}r\mathcal{K}(\Gamma)$; then, since $\mathcal{B}(x)$ is a separating collection (Lemma 6.5), it follows that

$$(7.1) \qquad\qquad \bigcup_{S:S\in\mathcal{B}(x)} S = N,$$

$$(7.2) \qquad\qquad \bigcap_{S:S\in\mathcal{B}(x)} S = \emptyset,$$

because $\mathcal{B}(x)$ is not empty and its members are proper non-empty subsets of N.

By applying Lemma 6.2 repeatedly to unions and intersections of members of $\mathcal{B}(x)$, one concludes that either there exist two coalitions S_1 and T_1 in $\mathcal{B}(x)$ such that $S_1 \cap T_1 = \emptyset$, or there exists two coalitions S_2 and T_2 in $\mathcal{B}(x)$ such that $S_2 \cup T_2 = N$. In view of the fact that $e(N, x) = e(\emptyset, x) = 0$, it follows from (2.8) and (5.6) that $e(S, x) \leq 0$ for every

coalition in Γ. Consequently, $x \in \mathcal{C}(\Gamma)$ (Definition 4.3).
This concludes the proof.

Theorem 7.2. <u>The kernel (for the grand coalition) of a
convex game consists of a single point</u>.

<u>Proof</u>. In view of Theorem 5.1 it is sufficient to prove
that the pre-kernel of a convex game consists of a unique
point. The theorem certainly holds for 1-person games.
We shall proceed by induction, assuming that Γ is an n-
person game, $n \geq 2$. Let $x, y \in \mathcal{Pr}\mathcal{K}(\Gamma)$. Denote

$$s(x) = e(S, x), \quad S \in \mathcal{B}(x),$$

$$s(y) = e(R, y), \quad R \in \mathcal{B}(y).$$

Without loss of generality we may assume that

$$(7.3) \qquad\qquad s(x) \leq s(y).$$

Since $\mathcal{B}(y)$ is a separating near-ring collection (Lemmas
6.2 and 6.5) which contains a nonempty subset of N, it
must contain a balanced collection $\rho = \{R_1, R_2, \ldots, R_d\}$
(Theorem 6.8). If $R_j \notin \mathcal{B}(x)$ then $e(R_j, x) < s(x) \leq s(y) =$
$e(R_j, y)$. Consequently, $x(R_j) > y(R_j)$. If $R_j \in \mathcal{B}(x)$ then

we can at least conclude that $x(R_j) \geq y(R_j)$. Multiplying these inequalities by the balancing coefficients and summing over j, we obtain $x(N) \geq y(N)$, with equality occuring only if $\mathcal{P} \subset \mathcal{B}(x)$ and $s(x) = s(y)$. But equality must occur because $x(N) = v(N) = y(N)$ (see (2.2)). All we need to conclude from this observation is that there exists a coalition R in $\mathcal{B}(x) \cap \mathcal{B}(y)$ and, moreover,

$$(7.4) \qquad x(R) = y(R), \quad x(N - R) = y(N - R).$$

Now let $m(x) + 1$ be the last stage of the profile of x and $m(y) + 1$ the last stage of the profile of y. Consider the stage games (T_R^*, v_R^*) and (T_{N-R}^*, v_{N-R}^*) as given in Corollary 5.8, with respect to x and $i = m(x) + 1$. Consider also the analogous stage games (T_R^{**}, v_R^{**}) and $(T_{N-R}^{**}, v_{N-R}^{**})$ with respect to y and $i = m(y) + 1$. The players in all four of these stage games are 1-element sets (see (3.11)).

Since x and y belong to $\mathcal{P}r\mathcal{K}(\Gamma)$, they __a fortiori__ belong to $\mathcal{2}(\Gamma)$ (Lemma 7.1). By Theorem 5.2, all the stage games are convex; therefore their kernels and pre-kernels coincide (Theorem 5.1). Rename the players, if necessary, so that $R = \{1, 2, \ldots, r\}$, then, by Theorem 4.6, we conclude that

$$(7.5) \qquad (x_1, x_2, \ldots, x_r) \in \text{Pr}\mathcal{K}(T_R^*, v_R^*),$$

$$(7.6) \qquad (x_{r+1}, x_{r+2}, \ldots, x_n) \in \text{Pr}\mathcal{K}(T_{N-R}^*, v_{N-R}^*),$$

$$(7.7) \qquad (y_1, y_2, \ldots, y_r) \in \text{Pr}\mathcal{K}(T_R^{**}, v_R^{**}), \text{ and}$$

$$(7.8) \qquad (y_{r+1}, y_{r+2}, \ldots, y_n) \in \text{Pr}\mathcal{K}(T_{N-R}^{**}, v_{N-R}^{**}).$$

Now, by (7.4) and Corollary 5.8, the games $(T_R^*; v_R^*)$ and $(T_R^{**}; v_R^{**})$ are __the same game__, because they have the same set of players and the same characteristic function. Similarly, (T_{N-R}^*, v_{N-R}^*) and $(T_{N-R}^{**}, v_{N-R}^{**})$ are the same game. Identical games possess the identical kernels. Since all of them have fewer than n players, then, by the induction hypothesis, their kernels (= pre-kernels) consist of single points. Consequently, by (7.5) - (7.8), we conclude that $x = y$. This completes the proof, in view of the fact that the kernel is known not be be empty (see [7]).

Theorem 7.2 brings to an end the main part in the study of the kernel for convex games. We know exactly its shape; namely--a point. There remains, however, the problem of locating this point; i.e., stating where it lies--preferably in geometrical terms. Fortunately, general theorems are

available in the literature which enable us to complete

this task: In [14], D. Schmeidler introduced the <u>nucleolus</u>

of a game and proved that it is a nonempty subset of the

kernel, consisting of a unique point.* In [9] we present

a characterization of the location of the nucleolus for a

general cooperative game. It turns out that the nucleolus

lies precisely at the, so called, <u>lexicographic center</u> of

the core; a point which, for games with a nonempty core,

lies in the relative interior of the core and occupies

there a central position.** Roughly speaking, the lexico-

graphic center is obtained by "pushing inward" at equal

ℓ_1-distances the hyperplanes which support the core, stop-

ping the push of each of them just short of causing the

inside to become empty. We refer the reader to [9] for

definitions and further details and summarize the results

which are relevant to the present study in:

<u>Corollary 7.3</u>. <u>For convex games, the kernel (for the grand</u>

<u>coaliton) and the nucleolus [the lexicographic center] co-</u>

<u>incide</u>.

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*See also [5].

**In [17] it is shown that the core of an indecomposable
n-person convex game is (n - 1)-dimensional, and that the
core of a decomposable n-person convex game is the (n - p)-
dimensional cartesian product of the cores of its minimal
(i.e., indecomposable) components, which are themselves convex.

8. THE BARGAINING SET $m_1^{(i)}$ OF A CONVEX GAME

Let x be an imputation in a game (N; v) (see (2.3)).
An <u>objection</u> of a player k against a player ℓ, with respect
to x, is a pair $(\hat{y}; C)$, where C is a coalition containing
player k and not containing player ℓ, $\hat{y}$ is a vector* whose
indices are the members of C, satisfying: $\hat{y}(C) = v(C)$ and
$\hat{y}_i > x_i$ for each i in C. A <u>counter objection</u> to this ob-
jection is a pair $(\hat{z}; D)$, where D is a coalition contain-
ing player ℓ and not containing player k and $\hat{z}$ is a vector
whose indices are members of D, satisfying $\hat{z}(D) = v(D)$,
$\hat{z}_i \geq \hat{y}_i$ for $i \in D \cap C$ and $\hat{z}_i \geq x_i$ for $i \in D - C$.

<u>Definition 8.1</u>. An imputation x is said to belong to the
<u>bargaining set</u> $m_1^{(i)}(\Gamma)$ (for the grand coalition),** if
for any objection of one player against another with
respect to x, there exists a counter objection.

Clearly, $m_1^i(\Gamma) \supset C(\Gamma)$, because if $x \in C(\Gamma)$, no objec-
tions are possible. In this section we shall show that

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*The circumflex reminds us that $\hat{y}$ is not an n-tuple.

**The definition can be extended to cover situations
in which coalition-structures other than the grand coali-
tion are being considered (see, e.g., [4], [10]. For in-
tuitive background and justification see [1]).

if Γ is convex then $m_1^{(i)}(\Gamma) = \mathcal{C}(\Gamma)$. Since $m_1^{(i)}(\Gamma) \supset \mathcal{K}(\Gamma)$ (see [3]), this result furnishes another proof of Lemma 7.1.

The proof will make use of a lemma concerning convex games which is of interest in itself. First a definition: the <u>monotonic cover</u>* of a game $(N; v)$ is the game $(N; \acute{v})$ defined by

$$(8.1) \qquad \acute{v}(S) = \max_{R \subset S} v(R), \qquad \text{all } S \subset N.$$

It is clear that the monotonic cover is indeed monotonic (see (2.7)), and that $\acute{v}(S) \geq v(S)$ for all $S \subset N$.

<u>Lemma 8.2</u>. <u>The monotonic cover of a convex game is convex</u>.

<u>Proof</u>. Let $(N; v)$ be convex, let S_1, $S_2 \subset N$, and let R_1, R_2 be such that $R_i \subset S_i$ and $v(R_i) = \acute{v}(S_i)$, $i = 1, 2$. Then

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*Called "monotonic closure" in [8].

$$\acute{v}(S_1) + \acute{v}(S_2) = v(R_1) + v(R_2)$$

$$\leqq v(R_1 \cup R_2) + v(R_1 \cap R_2)$$

$$\leqq \acute{v}(R_1 \cup R_2) + \acute{v}(R_1 \cap R_2)$$

$$\leqq \acute{v}(S_1 \cup S_2) + \acute{v}(S_1 \cap S_2).$$

Since also $\acute{v}(\emptyset) = v(\emptyset) = 0$, $(N; \acute{v})$ is convex, as claimed.

<u>Theorem 8.3</u>. <u>The bargaining set $m_1^{(i)}$ (for the grand coali-
tion) of a convex game coincides with the core of the game</u>.

<u>Proof</u>. Let $\Gamma \equiv (N; v)$ be a convex game, and let x be an
imputation with $x \notin \mathcal{C}(I)$. We must show that $x \notin m_1^{(i)}(\Gamma)$.
Let C be a maximal element of $\mathcal{B}(x)$, so that (see Lemma 5.6)

(8.2) $e(S, x) \leqq e(C, x)$ for all $S \subset N$,   and

(8.3) $e(S, x) < e(C, x)$ if $C \subset S \subset N$, $S \neq C$.

Write $e(S)$ for $e(S, x)$; then $(C; e)$ is a convex game, by
(5.6). Its monotonic cover $(C; \acute{e})$ is also convex, by
Lemma 8.1, and so has a nonempty core (see [17]). Take û

in that core, then

$$(8.4) \qquad \hat{u}(C) = \acute{e}(C) = \underset{R \subset C}{\text{Max}}\, e(R, x) = e(C, x) > 0$$

(see (2.2), (8.1) and (8.2)), and, for each $R \subset C$,

$$(8.5) \qquad \hat{u}(R) \geqq \acute{e}(R) \geqq e(R) = e(R, x)$$

(see (4.2) and (8.1)), and also, for each $i \in C$,

$$(8.6) \qquad \hat{u}_i \geqq \acute{e}(\{i\}) \geqq \acute{e}(\emptyset) = 0$$

(see (4.2), (8.1) and (5.1)). To construct the objection,
let $k \in C$ be such that $\hat{u}_k > 0$ and let $\ell \in N - C$ be arbitrary.
(This is possible because $\hat{u}(C) > 0$ and $C \neq \emptyset, N$.) Define

$$(8.7) \qquad \begin{cases} \hat{y}_i = x_i + \hat{u}_i + \epsilon, \text{ for } i \in C, i \neq k, \text{ and} \\ \hat{y}_k = x_k + \hat{u}_k - (c - 1)\epsilon, \end{cases}$$

where c is the size of C and ϵ satisfies $0 < (c - 1)\epsilon < u_k$.
Then it is clear, in view of (8.4) and (8.6), that $(\hat{y}; C)$
is an objection of k against ℓ with respect to x.

Now let D be any coalition containing ℓ but not k. By (8.5), (5.6) and (8.3) we have

$$\hat{u}(D \cap C) \geqq e(D \cap C, x) \geqq e(D, x) + e(C, x) - e(D \cup C, x)$$
(8.8)
$$> e(D, x)$$

(strict inequality because $\ell \in (D \cup C) - C$). Hence

$$v(D) = e(D, x) + x(D) < \hat{u}(D \cap C) + x(D) \leqq \hat{y}(D \cap C) + x(D - C),$$

using (8.7) and (8.8). This shows that D is not strong enough to support a counter objection. Hence "the objection is sustained," and $x \notin \mathcal{m}_1^{(i)}$, as was to be shown.

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